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SOLUTIONS TO ARBITRARILY ORIENTED PERIODIC DISLOCATION AND EIGENSTRAIN DISTRIBUTIONS IN A HALF-SPACE

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Abstract—A solution is given to the stress field produced by a periodic continuous distribution of dislocations (or plastic distortion) in a half-space, for the general case where the cartesian co-ordinate system specifying the distribution is arbitrarily oriented with respect to the free surface. When a plastic distortion is prescribed, the condition of full fixity at the surface is also dealt with. It is then shown how the theory may be applied to determine the stress and displacement fields in a fibre reinforced material. The problem of a Frank dislocation network inclined to the free surface of a half-space is finally considered.

1. INTRODUCTION

THE usefulness of the theory of continuously distributed dislocations in the analysis of problems in material behaviour is now generally accepted. For example, work hardening theories often depend on dislocation pile-up mechanisms which can be readily treated by a continuum approach, as first indicated by Leibfried [1]. Also much work has been done in recent years in establishing a correspondence between continuous dislocation theory and the mathematical theory of plasticity, a general formulation being provided by Mura [2, 3]. Such methods have been used by Bilby *et al.* [4, 5] who considered the yielding of cracks subjected to shear and by Owen [6] in the determination of elastic–plastic stress–strain relationships of composite materials.

The solution to a prescribed orthogonal periodic distribution of continuous dislocations or plastic distortion in an anisotropic unbounded medium has been given by Mura [7]. Such a result is of importance as solutions to other problems may then be readily generated by Fourier methods. This result has been extended by Owen and Mura [8] to the case of an isotropic half-space, the analysis being limited to the situation where the free surface is parallel to one of the co-ordinate planes prescribing the dislocation distribution.

However, in actual materials a free surface may have an arbitrary disposition with respect to a dislocation system; being dependent on the availability of slip and cleavage planes, etc. Therefore it is the aim of this paper to provide solutions for a half-space containing a dislocation or eigenstrain distribution in the form of a single exponential term described in a cartesian reference system which may be arbitrarily oriented with respect to the free surface. For the case of a prescribed plastic distortion (or eigenstrain) the condition of full fixity of the surface is also dealt with.

Much attention has been focussed on the analysis of fibre reinforced materials, a comprehensive study of the analytical solutions available having been made by Holister and Thomas [9]. Continuous fibres are readily analysed by simple mechanics, but discontinuous fibre systems, which are far more widely used, present theoretical difficulties due to the singularities at the fibre ends.

The results developed are then applied to the determination of the stress and displacement fields in such a composite material. Finally the problem of a Frank dislocation network inclined to the free surface of a half-space is investigated.

2. CONTINUOUS DISTRIBUTION OF DISLOCATIONS

Consider a material containing a large number of randomly oriented moving dislocations. It is often convenient to analyse such a system as a distribution of continuously distributed dislocations defined, with respect to a cartesian system x_i, by the following dislocation density and velocity tensors

$$\begin{aligned} \alpha_{hi} &= \sum n\nu_h b_i \\ V_{lhi} &= \sum n V_l \nu_h b_i \end{aligned} \tag{1}$$

where ν_h is the unit vector tangent to the dislocation line and n is the number of dislocations with direction ν_h, Burgers vector b_i and velocity V_l, crossing a unit area perpendicular to ν_h. The summation $\sum$ is taken with respect to all sets of dislocations with different values of n, ν_h, b_i and V_l at the point under consideration. The above two tensors are related, through the plastic distortion by

$$\alpha_{hi} = -\varepsilon_{hlk}\beta^*_{ki,l} \tag{2}$$

$$\dot{\beta}^*_{ki} = -\varepsilon_{kmn}V_{mni} \tag{3}$$

where ε_{hlk} is the unit permutation tensor and $,l$ denotes partial differentiation with respect to the co-ordinate x_l. The quantity β^*_{ki} is the plastic distortion and $\dot{\beta}^*_{kl}$ its time rate. The plastic strain or eigenstrain ε^*_{ki} is given by

$$\varepsilon^*_{ki} = \tfrac{1}{2}(\beta^*_{ki}+\beta^*_{ik}). \tag{4}$$

The plastic strain is the non-elastic deformation produced by the dislocation distribution. Physically this may be associated with the plastic deformation of materials or the "initial straining" (or eigenstrain) produced by lack of fit or thermal conditions.

The total strain is

$$\varepsilon_{ki} = \varepsilon^e_{ki}+\varepsilon^*_{ki} \tag{5}$$

and the elastic strain ε^e_{ki} is related to the stress by

$$\sigma_{pq} = C_{pqmn}\varepsilon^e_{mn}. \tag{6}$$

When a stationary dislocation distribution of the periodic form

$$\alpha_{hi} = \bar{\alpha}_{hi}\,\mathrm{e}^{ic_t x_t} \tag{7}$$

where $\bar{\alpha}_{hi}$ and c_i are arbitrary constants, exists in an unbounded medium the stress field is given by Mura [7] to be

$$\sigma^u_{pq} = ic_l\varepsilon_{njh}C_{pqmn}C_{ijkl}L_{km}\bar{\alpha}_{hi}\,\mathrm{e}^{ic_t x_t}. \tag{8}$$

For an isotropic material

$$L_{km} = \frac{\delta_{km}(\lambda+2\mu)c^2 - c_k c_m(\lambda+\mu)}{c^4\mu(\lambda+2\mu)} \tag{9}$$

where

$$c^2 = c_1^2 + c_2^2 + c_3^2$$

and

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} \tag{10}$$

δ_{ij} being the Kronecker delta and λ and μ are the usual Lamé constants.

For the case when a plastic distortion is prescribed as

$$\beta^*_{nm} = \bar{\beta}^*_{nm}\,\mathrm{e}^{ic_tx_t} \tag{11}$$

where $\bar{\beta}^*_{nm}$ is an arbitrary function of the constants c_i, the stress and displacement fields are respectively given by

$$\sigma^u_{pq} = -C_{pqmn}(-c_lc_nC_{ijkl}L_{km}\bar{\beta}^*_{ji} + \bar{\beta}^*_{nm})\,\mathrm{e}^{ic_tx_t} \tag{12}$$

and

$$U^u_a = -iC_{ijkl}c_lL_{ka}\bar{\beta}^*_{ji}\,\mathrm{e}^{ic_tx_t}. \tag{13}$$

3. ARBITRARILY ORIENTED PERIODIC DISLOCATION DISTRIBUTIONS IN A HALF-SPACE

Consider a half-space with free surface defined by $x'_1 = 0$ as shown in Fig. 1. Suppose that a continuous dislocation distribution exists within this half-space which is periodic with respect to a second arbitrarily oriented cartesian reference system, x_i. The relative orientation of the two sets of co-ordinate axes is defined by

$$\begin{aligned} x'_i &= a_{ij}x_j \\ x_i &= a_{ji}x'_j \end{aligned} \tag{14}$$

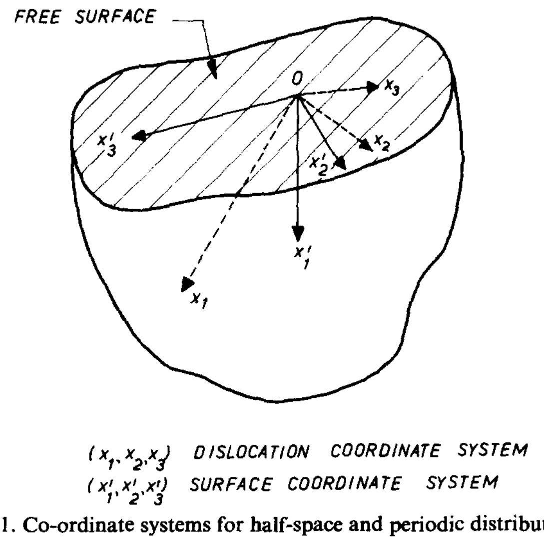

FIG. 1. Co-ordinate systems for half-space and periodic distributions.

where a_{ij} is the set of direction cosines for the two systems. Also, the transformation of stresses is given by

$$\sigma'_{ij} = a_{ir}a_{js}\sigma_{rs}. \tag{15}$$

It is required that all tractions on the free surface $x'_1 = 0$ vanish. The surface tractions are

$$X'_m = \sigma'_{m1}n'_1 = a_{mr}a_{1s}\sigma_{rs}n'_1 \tag{16}$$

where n'_1 is the unit normal to the free surface. Then from (8) and (16) the tractions at any point $(0, \mathbf{x}'_2, \mathbf{x}'_3)$ on the free surface are

$$X'_m = A'_m \, e^{ic_i\mathbf{x}_i} \tag{17}$$

with

$$A'_m = ia_{mr}a_{1s}\varepsilon_{njh}C_{rsun}C_{ijkl}L_{ku}\bar{\alpha}_{hi} \tag{18}$$

and the values $\mathbf{x}_i$ being the representation of the point $(0, \mathbf{x}'_2, \mathbf{x}'_3)$ in the x_i co-ordinate system.

Using (14) it may be shown that

$$c_ix_i = c'_ix'_i \tag{19}$$

where

$$c'_i = a_{ij}c_j \tag{20}$$

Using (19) in (17)

$$X'_m = A'_m \, e^{i(c'_2\mathbf{x}'_2 + c'_3\mathbf{x}'_3)}. \tag{21}$$

These free surface tractions may be eliminated by the superposition of forces $-X'_m$ to this surface. Defining $G'_{km}(x' - \mathbf{x}')$ as the displacement in the x'_k direction at the point (x'_1, x'_2, x'_3) due to a unit concentrated force in the x'_m direction acting at the point $(0, \mathbf{x}'_2, \mathbf{x}'_3)$ on the surface, the superimposed stress field due to $-X'_m$ is given on use of (6) and (21) to be

$$\sigma'^S_{pq} = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A'_m C_{pqkl}\frac{\partial G'_{km}}{\partial x'_l}\, e^{i(c'_2\mathbf{x}'_2 + c'_3\mathbf{x}'_3)}\, d\mathbf{x}'_2\, d\mathbf{x}'_3. \tag{22}$$

The Green's function, G'_{km}, for this problem is given by Owen and Mura [8] and the integration procedure for (22) is identical to that in Ref. [8], yielding

$$\sigma'^S_{pq} = -\frac{A'_m C_{pqkl}}{2\sqrt{(c'^2_2 + c'^2_3)}}\frac{\partial}{\partial x'_l}[I'_{km}\, e^{i(c'_2x'_2 + c'_3x'_3)}\, e^{-x'_1\sqrt{(c'^2_2 + c'^2_3)}}], \tag{23}$$

where

$$\begin{aligned} I'_{km} = \frac{1}{\mu}\Big\{&\delta_{km} - ix'_1[\delta_{1k}c'_m + \delta_{1m}c'_k - \delta_{1k}\delta_{1m}(c'_k + c'_m + i\sqrt{(c'^2_2 + c'^2_3)})] \\ &- \frac{ic'_kc'_m}{\sqrt{(c'^2_2 + c'^2_3)}}(\delta_{k2} + \delta_{k3})(\delta_{m2} + \delta_{m3}) + \frac{i}{c'^2_2 + c'^2_3}(c'_3\delta_{k2} - c'_2\delta_{k3})(c'_3\delta_{m2} - c'_2\delta_{m3})\Big\} \\ &+ \frac{1}{\lambda + \mu}\Big\{\delta_{k1}\delta_{m1} + \frac{c'_kc'_m}{c'^2_2 + c'^2_3}(1 - \delta_{k1}\delta_{m1}) + \frac{i}{\sqrt{(c'^2_2 + c'^2_3)}}(\delta_{m1}c'_k - \delta_{k1}c'_m)\Big\} \end{aligned} \tag{24}$$

and it is understood that $\sqrt{c_m'^2} = |c_m'|$. Substituting for A_m' in (23) from (18) and adding the whole space stress field (8) gives the complete stress field due to dislocation distribution (7), to be

$$\sigma'_{ab} = ic_s\varepsilon_{njh}C_{ijts}L_{tr}\bar{\alpha}_{hi}\, e^{i(c_2'x_2'+c_3'x_3')}\Bigg[a_{ap}a_{bq}C_{pqrn}\, e^{ic_1'x_1'} + \tfrac{1}{2}a_{mu}a_{1l}C_{ulrn}C_{abkw} \times \left\{\left(\frac{ic_2'\delta_{2w}+ic_3'\delta_{3w}}{\sqrt{(c_2'^2+c_3'^2)}} - \delta_{1w}\right)I'_{km} + \frac{\delta_{1w}}{\sqrt{(c_2'^2+c_3'^2)}}\frac{\partial I'_{km}}{\partial x_1'}\right\} e^{-x_1'\sqrt{(c_2'^2+c_3'^2)}}\Bigg]. \tag{25}$$

In the x_i reference system the stress components are

$$\sigma_{pq} = a_{ap}a_{bq}\sigma'_{ab}. \tag{26}$$

It is worth emphasising that in (25) the quantity L_{tr} contains c_i and not c_i'.

The stress field for a plastic distortion prescribed periodically in the x_i co-ordinate system in the form (11) follows the previous analysis with (18) replaced by

$$A_m' = -a_{mr}a_{1s}(-c_lc_nC_{ijkl}L_{ku}\beta^*_{ji} + \beta^*_{un})C_{rsun} \tag{27}$$

and (8) replaced by (12). This results in the following total stress field.

$$\sigma'_{ab} = (c_sc_nC_{ijts}L_{tr}\beta^*_{ji} - \beta^*_{nr})\, e^{i(c_2'x_2'+c_3'x_3')}\Bigg[a_{ap}a_{bq}C_{pqrn}\, e^{ic_1'x_1'} + \tfrac{1}{2}a_{mu}a_{1l}C_{ulrn}C_{abkw} \times \left\{\left(\frac{ic_2'\delta_{2w}+ic_3'\delta_{3w}}{\sqrt{(c_2'^2+c_3'^2)}} - \delta_{1w}\right)I'_{km} + \frac{\delta_{1w}}{\sqrt{(c_2'^2+c_3'^2)}}\frac{\partial I'_{km}}{\partial x_1'}\right\} e^{-x_1'\sqrt{(c_2'^2+c_3'^2)}}\Bigg]. \tag{28}$$

Using (21), the superimposed tractions $-X_m'$ produce displacements of

$$U_a'^s = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_m'G_{am}'\, e^{i(c_2'x_2'+c_3'x_3')}\, dx_2'\, dx_3'. \tag{29}$$

Substituting for A_m' from (27) and adding the whole space displacements (13) gives the displacements in the x_i' co-ordinate directions to be

$$U_a' = e^{i(c_2'x_2'+c_3'x_3')}\Bigg\{-ia_{au}c_sC_{ijts}L_{tu}\beta^*_{ji}\, e^{ic_1'x_1'} + \frac{a_{mr}a_{1s}C_{rsun}}{2\sqrt{(c_2'^2+c_3'^2)}} \times (-c_sc_nC_{ijts}L_{tu}\beta^*_{ji} + \beta^*_{un})I'_{am}\, e^{-x_1'\sqrt{(c_2'^2+c_3'^2)}}\Bigg\}. \tag{30}$$

4. PERIODIC PLASTIC DISTORTION DISTRIBUTION IN A HALF-SPACE WITH THE SURFACE RIGIDLY CONSTRAINED

Noting (19) and that,

$$U_a' = a_{ab}U_b \tag{31}$$

the whole space displacements at any point $(0, x_2', x_3')$ on the surface is given from (13) to be

$$U_m'^0 = B_m'\, e^{i(c_2'x_2'+c_3'x_3')} \tag{32}$$

where

$$B_m' = -ia_{mu}c_lC_{ijkl}L_{ku}\beta^*_{ji} \tag{33}$$

In order to satisfy the condition of zero displacements on the surface $x_1' = 0$, it is necessary to prescribe additional displacements of $-U_m'^0$. These give rise to additional stresses of

$$\sigma_{pq}'^s = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_m' C_{pqkl}\frac{\partial H_{km}'}{\partial x_l'} e^{i(c_2'x_2'+c_3'x_3')}\,dx_2'dx_3' \quad (34)$$

where H_{km}' is the displacement in the x_k' direction at the point (x_1', x_2', x_3') due to a prescribed unit displacement at $(0, x_2', x_3')$ in the x_m' direction, and is also recorded in Ref. [8]. Substituting in (34) for B_m' from (33) and adding the whole space stress field (12) gives the stress field in a half-space with rigidly constrained surface due to an arbitrarily oriented prescribed plastic distortion, to be

$$\sigma_{pq}' = e^{i(c_2'x_2'+c_3'x_3')}\Big\{a_{pu}a_{ql}C_{ulrn}(c_s c_n C_{ijts}L_{tr}\beta_{ji}^* - \beta_{nr}^*)\,e^{ic_1'x_1'}$$
$$+ a_{ma}C_{ijts}c_s L_{ta}\beta_{ji}^* C_{pqkl}\left[(\delta_{1l}.\, i\sqrt{(c_2'^2+c_3'^2)}+\delta_{2l}c_2'+\delta_{3l}c_3')J_{km}' - \delta_{1l}\frac{\partial J_{km}'}{\partial x_1'}\right] e^{-x_1'\sqrt{(c_2'^2+c_3'^2)}}\Big\} \quad (35)$$

where

$$J_{km}' = \delta_{km} + \frac{\lambda+\mu}{\lambda+3\mu}\Big\{\delta_{k1}\delta_{m1}\sqrt{(c_2'^2+c_3'^2)} - \frac{x_1'c_k'c_m'}{\sqrt{(c_2'^2+c_3'^2)}}(\delta_{m2}+\delta_{m3})(\delta_{k2}+\delta_{k3})$$
$$- ix_1'[\delta_{k1}c_m' + \delta_{m1}c_k' - \delta_{1k}\delta_{1m}(c_k'+c_m')]\Big\}. \quad (36)$$

The displacement field produced by displacements $-U_m'^0$ on the surface $x_1' = 0$ is

$$U_a'^s = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_m' H_{am}' e^{i(c_2'x_2'+c_3'x_3')}\,dx_2'\,dx_3'. \quad (37)$$

Substituting for B_m' from (33) and adding the whole space displacements (13) results in

$$U_a' = ic_l C_{ijkl}\beta_{ki}^* e^{i(c_2'x_2'+c_3'x_3')}\{a_{mu}L_{ku}J_{am}' e^{-x_1'\sqrt{(c_2'^2+c_3'^2)}} - a_{au}L_{ku}e^{ic_1'x_1'}\}. \quad (38)$$

5. SINGLE EDGE DISLOCATION IN A HALF-SPACE

To illustrate simply the use of the preceding theory, consider the two-dimensional problem of a discrete straight edge dislocation running in the x_3 direction at a distance a from the free surface and with arbitrary direction of Burgers vector, b, as shown in Fig. 2. As there is no variation in the x_3 direction, c_3 may be set equal to zero. The only non-zero component of the dislocation density tensor may be expressed as

$$\alpha_{31} = b\,.\,\delta(x_1 - a\cos\theta)\,.\,\delta(x_2 + a\sin\theta) \quad (39)$$

where δ is the Dirac delta function which may be expressed in Fourier form as

$$\delta(x_1-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ic_1(x_1-\alpha)}\,dc_1. \quad (40)$$

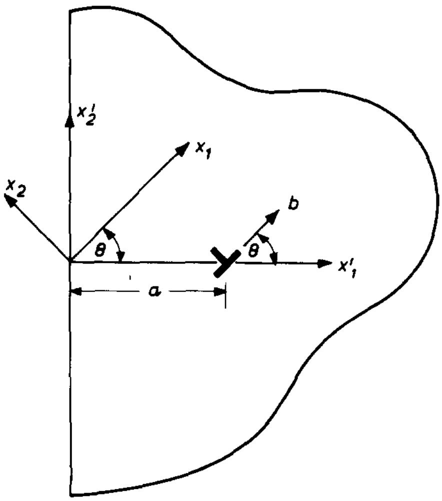

FIG. 2. Single edge dislocation in half-space with arbitrary direction of Burgers vector.

The set of direction cosines for this problem are

$$a_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{41}$$

From (25) the shear stress component σ'_{12} is found to be (for only α_{31} non-zero)

$$\sigma'_{12} = \frac{2i\mu\bar{\alpha}_{31}c_2\,e^{ic'_2x'_2}}{(1-\nu)(c_1^2+c_2^2)^2}\left\{\left[(c_2^2-c_1^2)\frac{\sin 2\theta}{2}-c_1c_2\cos 2\theta\right]e^{ic'_1x'_1}\right.$$
$$+\left[(c_2\sin\theta-c_1\cos\theta)(c_2\cos\theta+c_1\sin\theta)(x'_1|c'_2|-1)\right.$$
$$\left.\left.+ix'_1c'_2(c_1\sin\theta+c_2\cos\theta)^2\right]e^{-x'_1|c'_2|}\right\} \tag{42}$$

with ν being the Poisson's ratio of the material. Using (40) dislocation distribution (39) may be expressed in periodic form and substitution in (42), and noting (20), results in

$$\sigma'_{12} = \frac{i\mu b}{2\pi^2(1-\nu)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{c_2\,e^{ia(c_2\sin\theta-c_1\cos\theta)}}{(c_1^2+c_2^2)^2}\left\{(c_2^2-c_1^2)\frac{\sin 2\theta}{2}-c_1c_2\cos 2\theta\right\}$$
$$\times e^{i(c_1x_1+c_2x_2)}\,dc_1\,dc_2+\frac{\mu b}{2\pi^2(1-\nu)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}c_2\,e^{ia(c_2\sin\theta-c_1\cos\theta)}$$
$$\times\{-(c_1\sin\theta+c_2\cos\theta)^3x'_1+i(c_2\sin\theta-c_1\cos\theta)(c_2\cos\theta+c_1\sin\theta)(x'_1|c'_2|-1)\}$$
$$\times e^{-x'_1|c'_2|}\,e^{ic_2x_2}\,dc_1\,dc_2. \tag{43}$$

The first integral represents the whole space stress, the latter being the free surface contribution. The integrations in (43) may be performed by use of Cauchy's theorem in the complex

c_1 and c_2 planes, giving

$$\sigma'_{12} = \frac{\mu b}{2\pi(1-\nu)}\left\{\frac{(x'_1-a)[(x'_1-a)^2-x'^2_2]}{[(x'_1-a)^2+x'^2_2]^2}\cos\theta+\frac{x'_2[(x'_1-a)^2-x'^2_2]}{[(x_1-a)^2+x'^2_2]^2}\sin\theta\right.$$
$$-\left[\frac{2a(x'_1-a)(x'_1+a)^3+6x'_1(x'_1+a)x'^2_2-x'^4_2}{[(x'_1+a)^2+x'^2_2]^3}+\frac{(x'_1+a)[(x'_1+a)^2-x'^2_2]}{[(x'_1+a)^2+x'^2_2]^2}\right]\cos\theta$$
$$\left.+\left[\frac{4ax'_1x'_2[3(x'_1+a)^2-x'^2_2]}{[(x'_1+a)^2+x'^2_2]^3}-\frac{x'_2[(x'_1+a)^2-x'^2_2]}{[(x'_1+a)^2+x'^2_2]^2}\right]\sin\theta\right\}. \qquad (44)$$

Which agrees with the standard solution of Head [10].

6. APPLICATION TO THE ANALYSIS OF FIBRE REINFORCED MATERIALS

As previously stated a material reinforced by a system of discontinuous fibres presents analytical difficulties due to the geometric singularity at the fibre ends. However, a solution may be obtained by replacing the actual fibres by matrix material containing an initial strain distribution which produces the same restraint to deformation as the high modulus fibres.

Consider a half-space subjected to a tensile traction τ and reinforced by a regular fibre array as shown in Fig. 3. Conditions are assumed to be those of plane strain in the x_3 direction. The fibres all have a length/thickness ratio of a/b and the spacing is determined by the dimensions d and e. The position of the fibre array relative to the free surface is governed by f and h, while its orientation is defined by the angle θ. Suppose that the actual fibres are replaced by matrix material containing a constant initial strain, A, in the axial fibre directions only. It is assumed that the fibres have no lateral restraining effect and consequently the theory is restricted to fibres with a high length/thickness ratio. The value of A will be determined later. This prescribed plastic distortion may be expressed in the

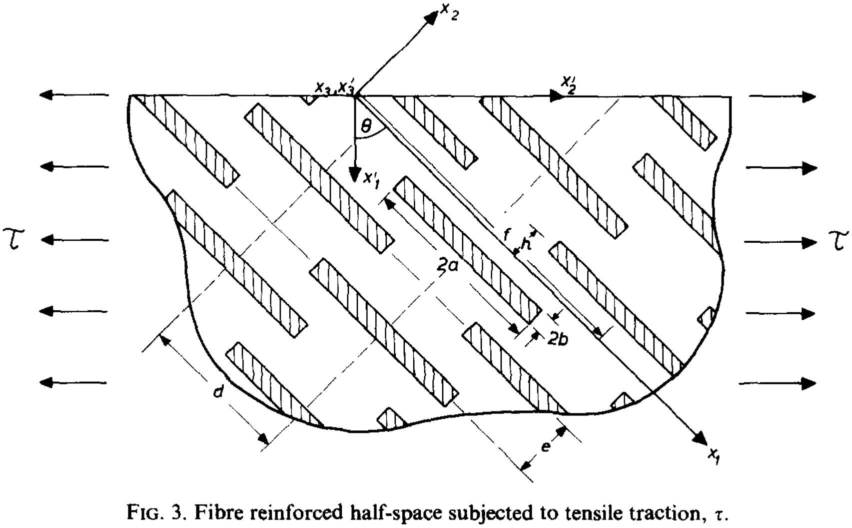

FIG. 3. Fibre reinforced half-space subjected to tensile traction, τ.

x_i co-ordinate system of Fig. 3, by means of Fourier series as follows

$$\beta^*_{11}(x_1, x_2) = \frac{2Aab}{de} + \frac{Ab}{\pi e} \sum_{\substack{p=-\infty \\ p \neq 0}}^{\infty} \frac{1}{p}\left(\sin\frac{p\pi a}{d} - \sin\frac{p\pi(d-a)}{d}\right) e^{(ip\pi/d)(x_1 - f)}$$
$$+ \frac{Aa}{\pi d} \sum_{\substack{q=-\infty \\ q \neq 0}}^{\infty} \frac{1}{q}\left(\sin\frac{q\pi b}{e} - \sin\frac{q\pi(e-b)}{e}\right) e^{(iq\pi/e)(x_2 - h)}$$
$$+ \frac{A}{\pi^2} \sum_{\substack{p=-\infty \\ p \neq 0}}^{\infty} \sum_{\substack{q=-\infty \\ q \neq 0}}^{\infty} \frac{1}{pq}\left(\sin\frac{p\pi a}{d}\sin\frac{q\pi b}{e}\right.$$
$$\left. + \sin\frac{p\pi(d-a)}{d}\sin\frac{q\pi(e-b)}{e}\right) e^{(ip\pi/d)(x_1 - f)} e^{(iq\pi/e)(x_2 - h)}. \tag{45}$$

All other components of β^*_{ij} being zero.

For a β^*_{11} of the form given by (11) the stresses and displacements in an unbounded medium, with respect to co-ordinate system x_i, are given by (12) and (13) as

$$\begin{aligned}
\sigma^u_{11} &= -\frac{2\mu}{1-\nu}\frac{c_2^4}{(c_1^2+c_2^2)^2}\beta^*_{11}\, e^{i(c_1x_1+c_2x_2)} \\
\sigma^u_{22} &= -\frac{2\mu}{1-\nu}\frac{c_1^2c_2^2}{(c_1^2+c_2^2)^2}\beta^*_{11}\, e^{i(c_1x_1+c_2x_2)} \\
\sigma^u_{12} &= \frac{2\mu}{1-\nu}\frac{c_1c_2^3}{(c_1^2+c_2^2)^2}\beta^*_{11}\, e^{i(c_1x_1+c_2x_2)} \\
U^u_1 &= \frac{-ic_1}{(c_1^2+c_2^2)^2}\left(\frac{2-\nu}{1-\nu}c_2^2 + c_1^2\right)\beta^*_{11}\, e^{i(c_1x_1+c_2x_2)} \\
U^u_2 &= \frac{-ic_2}{(c_1^2+c_2^2)^2}\left(\frac{\nu}{1-\nu}c_2^2 - c_1^2\right)\beta^*_{11}\, e^{i(c_1x_1+c_2x_2)}
\end{aligned} \tag{46}$$

where μ and ν are the material properties of the matrix material. Each term in (45) is of the form in (11), hence the stress and displacement fields can be found from (46) on substitution of appropriate values for c_1 and c_2, and summing over p and q, giving

$$\begin{aligned}
\sigma^u_{11} &= -\frac{4\mu Aa}{\pi(1-\nu)d}\sum_{q=1}^{\infty}\frac{1}{q}\left(\sin\frac{q\pi b}{e} - \sin\frac{q\pi(e-b)}{e}\right)\cos\frac{q\pi}{e}(x_2-h) \\
&\quad - \frac{8\mu A}{\pi^2(1-\nu)}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} Q\cdot\frac{q^3}{p}\cos\frac{p\pi}{d}(x_1-f)\,.\cos\frac{q\pi}{e}(x_2-h) \\
\sigma^u_{22} &= -\frac{8\mu A}{\pi^2(1-\nu)}\left(\frac{e}{d}\right)^2\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} Q\,.\,pq\cos\frac{p\pi}{d}(x_1-f)\,.\cos\frac{q\pi}{e}(x_2-h) \\
\sigma^u_{12} &= -\frac{8\mu A}{\pi^2(1-\nu)}\cdot\frac{e}{d}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} Q\,.\,q^2\sin\frac{p\pi}{d}(x_1-f)\,.\sin\frac{q\pi}{e}(x_2-h) \\
\frac{U^u_1}{a} &= \frac{4Ae^2}{\pi^3 ad}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} Q\cdot\left(\frac{2-\nu}{1-\nu}q^2 + \frac{e^2}{d^2}\cdot p^2\right)\sin\frac{p\pi}{d}(x_1-f)\,.\cos\frac{q\pi}{e}(x_2-h) \\
\frac{U^u_2}{b} &= \frac{4Ae}{\pi^3 b}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} Q\cdot\left(\frac{\nu}{1-\nu}q^2 - \frac{e^2}{d^2}\cdot p^2\right)\cos\frac{p\pi}{d}(x_1-f)\sin\frac{q\pi}{e}(x_2-h)
\end{aligned} \tag{47}$$

where

$$Q = \frac{\left(\sin\frac{p\pi a}{d}\sin\frac{q\pi b}{e}+\sin\frac{p\pi(d-a)}{d}\sin\frac{q\pi(e-b)}{e}\right)}{\left(q^2+\frac{e^2}{d^2}\cdot p^2\right)^2}. \tag{48}$$

It should be noted that all stresses and displacements are real.

The stress and displacement fields caused by the presence of the free surface may be found with the aid of equations (28) and (30) in which the terms dependent on I'_{km} represent the surface effects. For only β^*_{11} non-zero and of the form given by (11), the stresses and displacements (with respect to the x'_i co-ordinate system) due to the presence of the free surface can be shown to be

$$\begin{aligned}
\sigma'^s_{11} &= 2\mu S\{c'_2(1+x'_1|c'_2|)+(c_1\cos\theta-c_2\sin\theta)ix'_1c'_2\}\\
\sigma'^s_{22} &= 2\mu S\left\{c'_2(1-x'_1|c'_2|)+i\left(\frac{2c'_2}{|c'_2|}-x'_1c'_2\right)(c_1\cos\theta-c_2\sin\theta)\right\}\\
\sigma'^s_{12} &= 2\mu S\{-ix'_1c'^2_2+(x'_1|c'_2|-1)(c_1\cos\theta-c_2\sin\theta)\}\\
U'^s_1 &= \frac{S}{|c'_2|}\left\{c'_2(x'_1|c'_2|+2(1-\nu))-i\left(x'_1c'_2+\frac{c'_2}{|c'_2|}(1-2\nu)\right)(c_1\cos\theta-c_2\sin\theta)\right\}\\
U'^s_2 &= \frac{S}{|c'_2|}\left\{ic'_2\left(\frac{c'_2}{|c'_2|}(1-2\nu)-x'_1c'_2\right)+(2(1-\nu)-x'_1|c'_2|)(c_1\cos\theta-c_2\sin\theta)\right\}
\end{aligned} \tag{49}$$

where

$$S = \frac{c'_2c_2^2\beta^*_{11}}{(1-\nu)(c_1^2+c_2^2)^2}\,e^{-x'_1|c'_2|}\,e^{ic'_2x'_2}. \tag{50}$$

For this problem the transformation direction cosines are again given by (41) and hence from (20)

$$\begin{aligned}
c'_1 &= c_1\cos\theta-c_2\sin\theta\\
c'_2 &= c_1\sin\theta+c_2\cos\theta.
\end{aligned} \tag{51}$$

Once again since each term in (45) is of periodic form, summations over p and q yield the stresses and displacements due to the presence of the free surface to be

$$\begin{aligned}
\sigma'^s_{11} = {} & \frac{4\mu Aa\cos^2\theta}{\pi d(1-\nu)}\sum_{q=1}^{\infty}T.\left[(1+x'_1m)\cos\phi+x'_1\frac{q\pi}{e}\sin\theta\,.\sin\phi\right]\\
& +\frac{4\mu A}{\pi^2(1-\nu)}\sum_{\substack{p=-\infty\\p\neq 0}}^{\infty}\sum_{q=1}^{\infty}W.\left[(1+x'_1n)\cos\gamma-\left(\frac{p\pi}{d}\cos\theta-\frac{q\pi}{e}\sin\theta\right)x'_1\sin\gamma\right]\\
\sigma'^s_{22} = {} & \frac{4\mu Aa\cos^2\theta}{\pi d(1-\nu)}\sum_{q=1}^{\infty}T.\left[(1-x'_1m)\cos\phi+\left(\frac{2}{m}-x'_1\right)\frac{q\pi}{e}\sin\theta\sin\phi\right]\\
& +\frac{4\mu A}{\pi^2(1-\nu)}\sum_{\substack{p=-\infty\\p\neq 0}}^{\infty}\sum_{q=1}^{\infty}W.\left[(1-x'_1n)\cos\gamma-\left(\frac{p\pi}{d}\cos\theta-\frac{q\pi}{e}\sin\theta\right)\left(\frac{2}{n}-x'_1\right)\sin\gamma\right]
\end{aligned} \tag{52}$$

$$\sigma'^{s}_{12} = \frac{4\mu A a \cos^2\theta}{\pi d(1-\nu)} \sum_{q=1}^{\infty} T . \left[x'_1 \cos^2\theta . \frac{q\pi}{e} \sin\phi + (1 - x'_1 m) \sin\theta \cos\phi \right]$$

$$+ \frac{4\mu A}{\pi^2(1-\nu)} \sum_{\substack{p=-\infty \\ p\neq 0}}^{\infty} \sum_{q=1}^{\infty} W . \left[x'_1 \left(\frac{p\pi}{d} \sin\theta + \frac{q\pi}{e} \cos\theta \right) \sin\gamma \right.$$

$$\left. + \frac{\left(\frac{p\pi}{d} \cos\theta - \frac{q\pi}{e} \sin\theta \right)}{\left(\frac{p\pi}{d} \sin\theta + \frac{q\pi}{e} \cos\theta \right)} (x'_1 n - 1) \cos\gamma \right]$$

$$\frac{U'^{s}_1}{a} = \frac{2A \cos^2\theta}{\pi d(1-\nu)} \sum_{q=1}^{\infty} T . \left[(2(1-\nu) + x'_1 m) \cos\phi - \left(x'_1 + \frac{(1-2\nu)}{m} \right) \frac{q\pi}{e} \sin\theta \sin\phi \right] \Big/ m$$

$$+ \frac{2A}{\pi^2 a(1-\nu)} \sum_{\substack{p=-\infty \\ p\neq 0}}^{\infty} \sum_{q=1}^{\infty} W . \left[(x'_1 n + 2(1-\nu)) \cos\gamma \right.$$

$$\left. + \left(\frac{p\pi}{d} \cos\theta - \frac{q\pi}{e} \sin\theta \right) \left(x'_1 + \frac{(1-2\nu)}{n} \right) \sin\gamma \right]$$

$$\frac{U'^{s}_2}{b} = \frac{2Aa \cos^2\theta}{\pi b d(1-\nu)} \sum_{q=1}^{\infty} T . \left[\left(x'_1 \cos^2\theta - \frac{(1-2\nu)}{m} \right) \frac{q\pi}{e} \sin\phi + (x'_1 m - 2(1-\nu)) \sin\theta \cos\phi \right] \Big/ m$$

$$+ \frac{2A}{\pi^2 b(1-\nu)} \sum_{\substack{p=-\infty \\ p\neq 0}}^{\infty} \sum_{q=1}^{\infty} W . \left[\left(x'_1 - \frac{(1-2\nu)}{n} \right) \left(\frac{p\pi}{d} \sin\theta + \frac{q\pi}{e} \cos\theta \right) \sin\gamma \right.$$

$$\left. + \frac{\left(\frac{p\pi}{d} \cos\theta - \frac{q\pi}{e} \sin\theta \right)}{\left(\frac{p\pi}{d} \sin\theta + \frac{q\pi}{e} \cos\theta \right)} (2(1-\nu) - x'_1 n) \cos\gamma \right]$$

where

$$T = \frac{1}{q} \left(\sin\frac{q\pi b}{e} - \sin\frac{q\pi(e-b)}{e} \right) e^{-x'_1 m}$$

$$W = \frac{1}{pq} \left(\sin\frac{p\pi a}{d} \sin\frac{q\pi b}{e} + \sin\frac{p\pi(d-a)}{d} \sin\frac{q\pi(e-b)}{e} \right) \left(\frac{q\pi}{e} \right)^2 \frac{\left(\frac{p\pi}{d} \sin\theta + \frac{q\pi}{e} \cos\theta \right)^2}{\left(\left(\frac{p\pi}{d} \right)^2 + \left(\frac{q\pi}{e} \right)^2 \right)^2} e^{-x'_1 n}$$

$$m = \left| \frac{q\pi}{e} \cos\theta \right|$$

$$n = \left| \frac{p\pi}{d} \sin\theta + \frac{q\pi}{e} \cos\theta \right| \qquad (53)$$

$$\phi = \frac{q\pi}{e} (x'_2 \cos\theta - h)$$

$$\gamma = \frac{p\pi}{d} (x'_2 \sin\theta - f) + \frac{q\pi}{e} (x'_2 \cos\theta - h).$$

Once again these final expressions are entirely real. The complete stress and displacement fields are given by the sum of (47) and (52) (after transformation to a common co-ordinate system) and addition of the values associated with the applied traction, τ.

The value of the initial strain, A, may be determined from the condition that the fibres behave linearly over a specified fraction ρa of their length. Due to the applied stress τ the strain in the x_1 direction in the absence of any reinforcing fibres is

$$\varepsilon_{11} = \frac{\tau}{E}(\sin^2\theta - \nu\cos^2\theta)$$

where E and ν are the elastic properties of the matrix material. Due to the eigenstrain distribution there will be an additional strain along the axis of the fibre of

$$A\frac{U_1^u(\rho a, 0)}{\rho a}$$

since the displacement along a fibre is linear over a fraction ρa of its length. U_1^u is the displacement in the x_1 direction in the whole space due to a unit eigenstrain, and is given from (47). Therefore the total strain along a length ρa of the fibre is of constant amount

$$\frac{\tau}{E}(\sin^2\theta - \nu\cos^2\theta) + A\frac{U_1^u(\rho a, 0)}{\rho a}. \tag{54}$$

The total stress in the x_1 direction along the axis of a fibre is

$$\tau\sin^2\theta + A\sigma_{11}^u(0, 0)$$

where σ_{11}^u is the whole space stress, (47), due to a unit eigenstrain. The first stress term is that due to the applied load, the second being that produced by the eigenstrain distribution. The value of σ_{11}^u at $x_1 = 0$ may be taken as the stress is constant (see Fig. 7) along the effective length of the fibre. If the fibres have an elastic modulus of KE, where K is a constant, then the strain along the effective length ρa of the fibre is of constant value

$$\frac{1}{KE}(\tau\sin^2\theta + A\sigma_{11}^u(0, 0)) \tag{55}$$

Equating (54) and (55) gives the required value of A to be

$$A = \frac{\tau}{E}\frac{\left[\left(\frac{1}{K} - 1\right)\sin^2\theta + \nu\cos^2\theta\right]}{\left[\frac{U_1^u(\rho a, 0)}{\rho a} - \frac{\sigma_{11}^u(0, 0)}{KE}\right]}. \tag{56}$$

Numerical results were obtained, by the use of a computer, for a fibre orientation, θ, of 45° and a value of $K = 22$, being representative of a carbon fibre reinforced composite. The distance between centres of fibres was chosen as ten times the fibre thickness and the fibre length to thickness ratio taken as 24. The Poisson's ratio of the matrix material was assumed to be 0·45 being representative of an epoxy-resin commonly used. The displacement along the axis of a general fibre is shown in Fig. 4 for the case of an unbounded medium subjected to an arbitrary tensile stress τ. The additional displacements caused by the presence of a free surface at $x_1' = 0$ are shown in Fig. 5, as well as the total axial displacement.

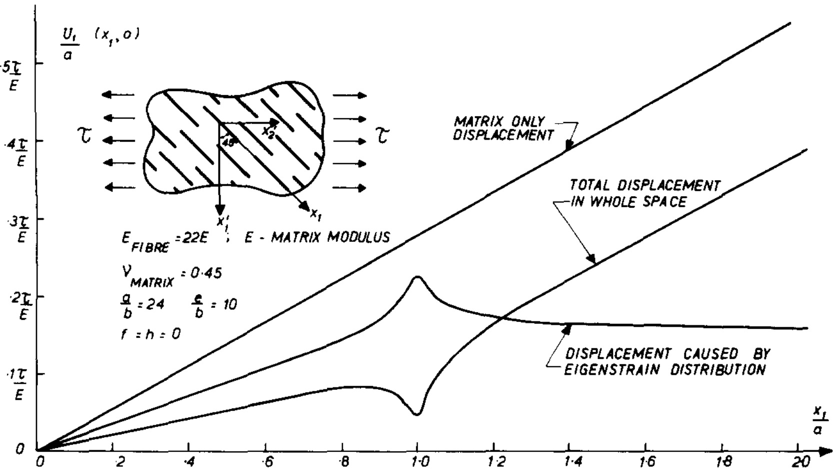

FIG. 4. Axial displacements of a typical fibre in whole space.

Figure 6 illustrates the shear stress distribution along the fibre–matrix interface for both whole space and half-space situations. The normal stress distribution along the axis of the fibre is shown in Fig. 7.

7. FRANK DISLOCATION NETWORK INCLINED TO A FREE SURFACE

Consider a regular Frank dislocation network of edge length $2a$ as shown in Fig. 8. For a perfectly hexagonal network every dislocation should be a pure screw according to

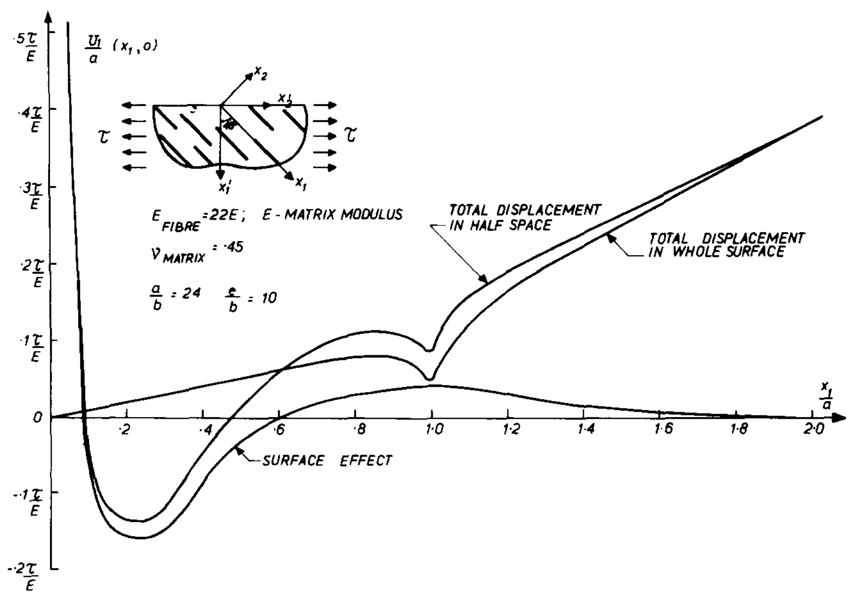

FIG. 5. Axial displacement of fibre in a half-space.

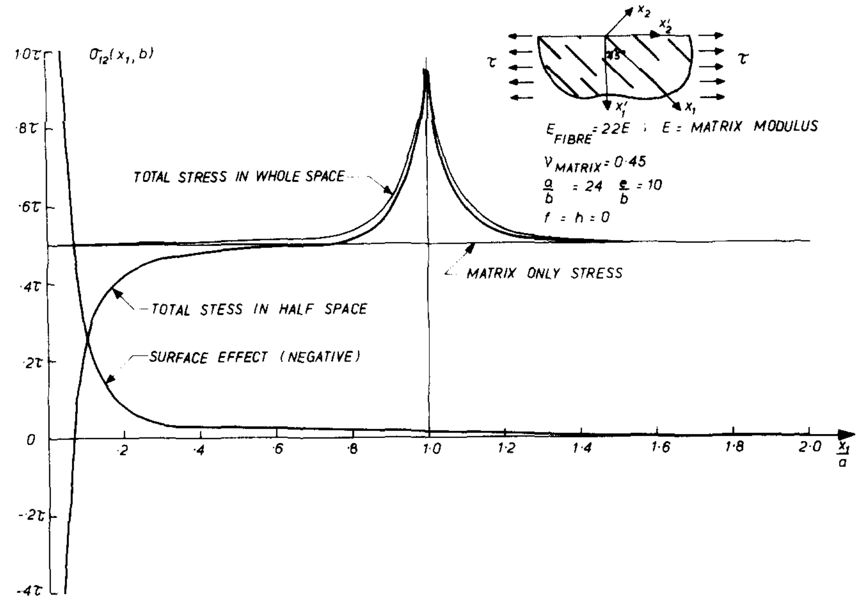

FIG. 6. Variation of shear stress σ_{12} along fibre/matrix interface of fibre in a reinforced half-space.

Frank [11]. Suppose such a network occurs in the vicinity of a free surface defined by $x_1' = 0$; the relative orientation of the surface and network being governed by the set of transformation direction cosines, a_{ij}, between the co-ordinate systems x_i and x_i'. The complete network is composed of three systems of dislocation segments parallel to AB, BC and AF and denoted by the superscripts 1, 2 and 3, respectively.

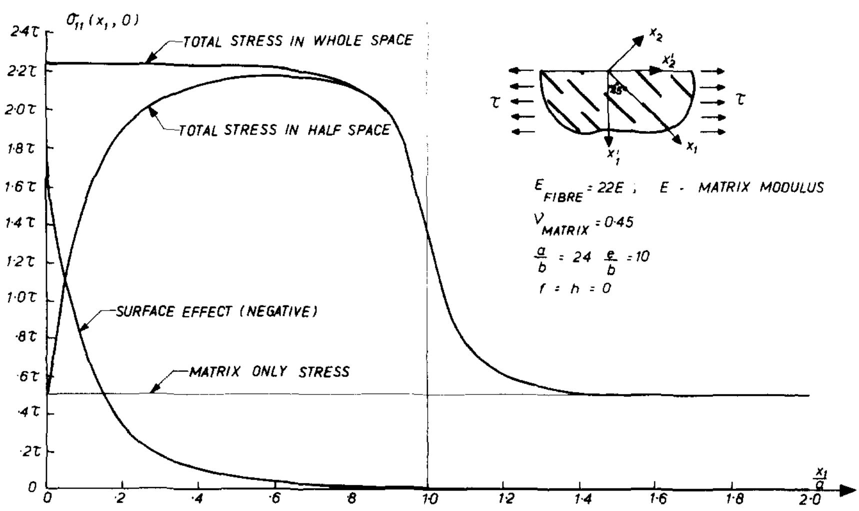

FIG. 7. Variation of normal stress σ_{11} along axis of fibre in a reinforced half-space.

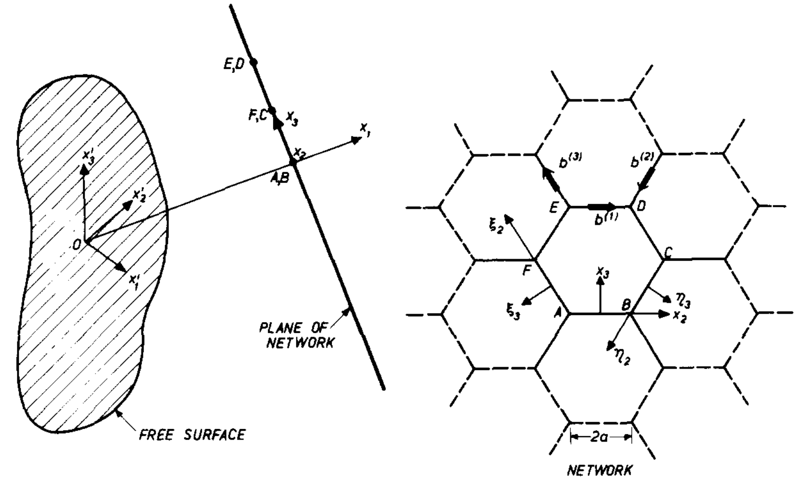

FIG. 8. Illustration of Frank dislocation network in a half-space showing surface and network co-ordinate systems.

The dislocation density of system 1 can be expressed in the x_i co-ordinate system in terms of Fourier series and integrals to be

$$\alpha_{22}^{(1)} = \frac{b^{(1)}}{4\pi\sqrt{(3)}a}\int_{-\infty}^{\infty}\left\{\frac{2}{3}+\frac{1}{3}\sum_{\substack{m=-\infty\\m\neq 0}}^{\infty}(1+\cos m\pi)\,e^{[im\pi/\sqrt{(3)}a]x_3}+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{1}{n\pi}\left[\sin\frac{n\pi}{3}\right.\right.$$
$$\left.+i\cos n\pi - i\cos\frac{2n\pi}{3}-\sin\frac{2n\pi}{3}\right]e^{(in\pi/3a)x_2}+\frac{1}{\pi}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\sum_{\substack{m=-\infty\\m\neq 0}}^{\infty}\frac{1}{n}\left[\sin\frac{n\pi}{3}\right. \tag{57}$$
$$\left.\left.+\cos n\pi\left(-i\cos\frac{2n\pi}{3}-\sin\frac{2n\pi}{3}+i\cos n\pi\right)\right]e^{(in\pi/3a)x_2}\,e^{[im\pi/\sqrt{(3)}a]x_3}\right\}e^{ic_1(x_1-d)}\,dc_1$$

all other components being zero and $b^{(1)}$ the Burgers' vector. Identical expressions hold for systems 2 and 3 but with x_2, x_3 replaced by η_2, η_3 and ξ_2, ξ_3, respectively.

The stress field for such a network parallel to the free surface of a half space has been determined by Owen and Mura [8], the results being presented as the stresses for such a network in an unbounded medium on which were superimposed the stresses caused by the free surface. For the present case the stress field for the network in an unbounded medium will be unchanged, and it only remains to find the portion of the stress field due to the presence of the free surface. For α_{22} of the form (7) the surface effect stresses can be expressed from (25), with respect to the x_i' co-ordinate system, as

$$\sigma_{11}'^{s} = K_u\{R_{22}x_1'(a_{2u}c_2'+a_{3u}c_3')-S_{22}a_{1u}(1+x_1'p)\}$$
$$\sigma_{22}'^{s} = \frac{K_u}{p}\left\{R_{22}\left[a_{2u}c_2'\left(2+\frac{2\nu c_3'^2}{p^2}-\frac{x_1'c_2'^2}{p}\right)+a_{3u}c_3'\left(\frac{2\nu c_3'^2}{p^2}-\frac{x_1'c_2'^2}{p}\right)\right]\right.$$
$$\left.-S_{22}a_{1u}\left(\frac{c_2'^2}{p}+\frac{2\nu c_3'^2}{p}-x_1'c_2'^2\right)\right\} \tag{58}$$

$$\sigma'^{s}_{33} = \frac{K_u}{p}\left\{R_{22}\left[a_{2u}c'_2\left(\frac{2\nu c'^2_2}{p^2}-\frac{x'_1c'^2_3}{p}\right)+a_{3u}c'_3\left(2+2\nu c'^2_2-\frac{x'_1c'^2_3}{p}\right)\right]\right.$$
$$\left.-S_{22}a_{1u}\left(\frac{c'^2_3}{p}+\frac{2\nu c'^2_2}{p}-x'_1c'^2_3\right)\right\} \qquad (58\text{ ctd.})$$
$$\sigma'^{s}_{12} = K_u\left\{R_{22}a_{1u}x'_1c'_2+S_{22}\left[a_{2u}\left(\frac{x'_1c'^2_2}{p}-1\right)+a_{3u}\frac{x'_1c'_2c'_3}{p}\right]\right\}$$
$$\sigma'^{s}_{13} = K_u\left\{R_{22}a_{1u}x'_1c'_3+S_{22}\left[a_{2u}\frac{x'_1c'_2c'_3}{p}+a_{3u}\left(\frac{x'_1c'^2_3}{p}-1\right)\right]\right\}$$
$$\sigma'^{s}_{23} = \frac{K_u}{p}\left\{R_{22}\left[a_{2u}c'_3\left(\frac{c'^2_3}{p^2}-\frac{x'_1c'^2_2}{p}\right)+a_{3u}c'_2\left(\frac{c'^2_2}{p^2}-\frac{x'_1c'^2_3}{p}\right)\right]\right.$$
$$\left.+S_{22}a_{1u}c'_2c'_3\left(x'_1-\frac{(1-2\nu)}{p}\right)\right\}$$

where

$$p = \sqrt{(c'^2_2+c'^2_3)}$$
$$K_u = -\varepsilon_{nj2}C_{2jts}a_{1l}C_{ulrn}c_sL_{tr} \qquad (59)$$

and

$$\bar{\alpha}_{22} = R_{22}+iS_{22} \qquad (60)$$

where both R_{22} and S_{22} are real.

Each term in (57) is of the periodic form (7), hence the stresses due to the presence of the free surface may be obtained from (58) on integrating with respect to c_1 from $-\infty$ to ∞ and performing the various summations over m and n. A computer program was written for this purpose, the contributions of systems 2 and 3 also being calculated and added. Numerical values were produced for the configuration shown in Fig. 9, for an arbitrary Burgers vector, b. Some typical stress components, expressed in the x'_i co-ordinate system, calculated along the line, HJ, defined by $x_2 = 0$, $x_3 = \sqrt{(3)}a/4$ are shown in Figs. 10–12.

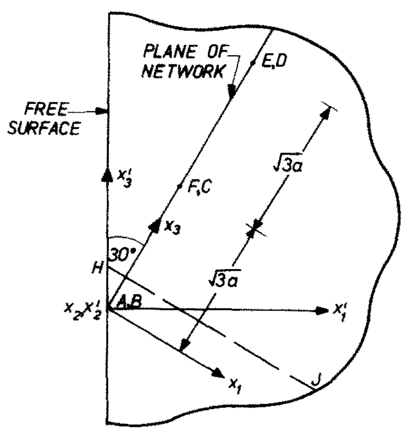

FIG. 9. Frank dislocation network in a half-space, inclined at 30° to the free surface.

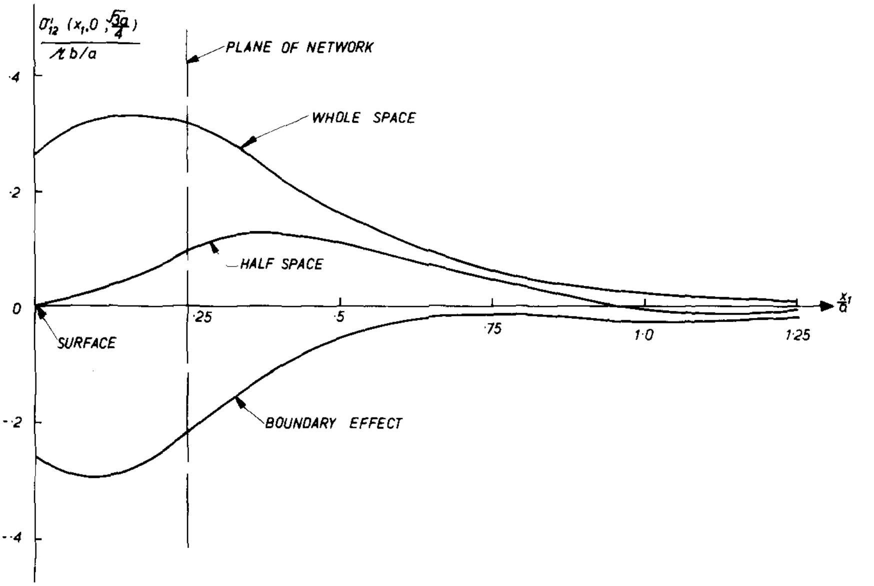

FIG. 10. Variation of stress component σ'_{12} along line HJ, defined by $x_2 = 0$, $x_3 = \sqrt{3}a/4$.

8. DISCUSSION OF RESULTS AND CONCLUSIONS

Solutions are given for the stress fields in a half-space due to a periodically prescribed distribution of dislocations and plastic distortion. In particular any arbitrary relative orientation between the free surface and the cartesian co-ordinate system which defines

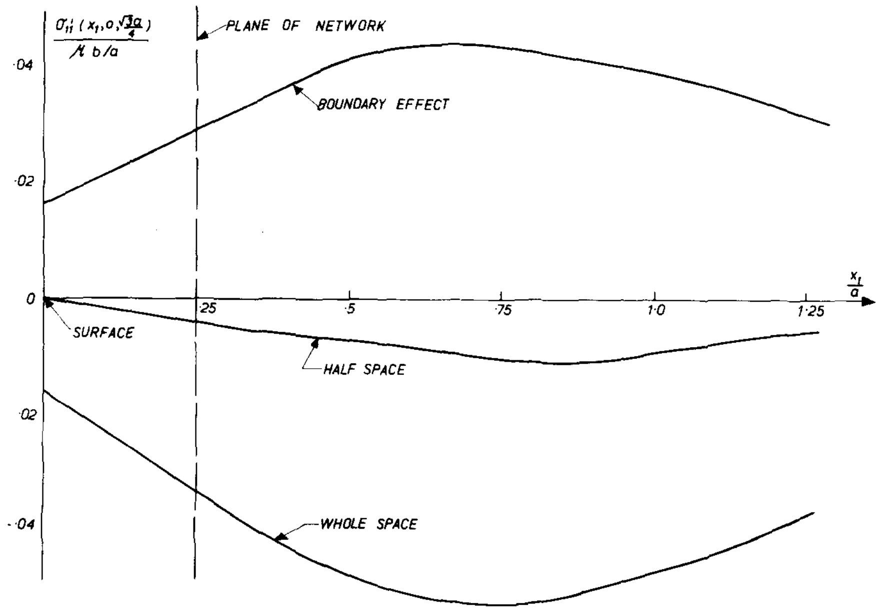

FIG. 11. Variation of stress component σ'_{11} along line HJ, defined by $x_2 = 0$, $x_3 = \sqrt{3}a/4$.

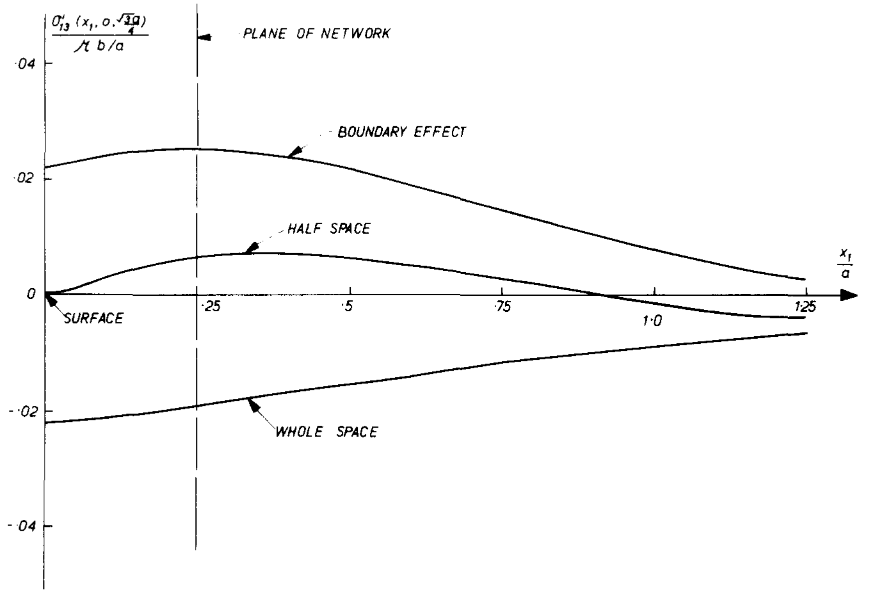

Fig. 12. Variation of stress component σ'_{13} along line HJ, defined by $x_2 = 0$, $x_3 = \sqrt{3}a/4$.

the distributions is permitted. For the case of prescribed plastic distortion the condition of complete fixity of the surface is also considered.

The theory is then applied to the problem of a material reinforced by a regular array of fibres. Figure 4 clearly demonstrates the restraint to deformation provided by the fibres, whilst a marked reduction in displacement near the fibre end is also evident. This latter effect is observed in practice and is due to the fact that full transfer of load from matrix to fibre only takes place after a certain distance from the fibre end. The main feature of the half space solution of Fig. 5 is the large deformation produced at the free surface.

The expected singularity in interfacial shear stress at the fibre end caused by the fibre geometry is evident in Fig. 6. A similar singularity occurs at $x_1 = 0$ due to the presence of the free surface. Figure 7 illustrates the basic load-bearing mechanism of fibre reinforced materials. It is seen that the fibre is highly stressed in the axial direction, the load being rapidly shed beyond the fibre end. The presence of the free surface, $x'_1 = 0$ merely results in the reduction of the fibre axial stress in this region.

For the Frank dislocation network problem, it is seen from Figs. 10–12 that the boundary stress requirements (i.e. $\sigma'_{1j} = 0$) are completely satisfied. A considerable reduction in stress values due to the presence of the free surface is also evident.

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Абстракт—Даётся решение для иоля напряжений, вызванного периодическим сплошным распределением дислокаций/или пластических дисторсий/в полуплоскости, для общего случая, в котором система декартовых координат, определяющая это распределение, произвольно направлена по отношении к свободной поверхности. Рассматривается также условие полной неподвижности на поверхности для пластической дисторсии.

Указывается, затем, способ применения теории для определения полей напряжений и перемещений в материале, усиленным волокном. Исследуется, наконец, задача соти дислокаций франка, наклонённых к свободной поверхности.